Dispersion properties of a circularly polarized electromagnetic wave in a collisional warm plasma

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Abstract :

In this article, a review of the work of M.L.Sawley [1] is done to study the propagation characteristics of a circularly polarized electromagnetic wave in a collisional warm plasma. The analytic expression for the linear dispersion relation is obtained with all the steps in detail, and analysed.

Keywords: Collisional Warm Plasma, Electromagnetic Wave, LCP Wave, RCP Wave

1 Introduction:

1.1 Basic concepts of plasma:

A plasma can be defined as an ionised gas of charged and neutral particles which satisfies the following criteria:

- It exhibits collective behaviour.
- The length scale of excitation is much larger than the Debye length.
- The frequency of oscillation of a charged particle is larger than the collisional frequency between them.

1.2 Occurence of plasma in nature:

It has often been said that 99% of the matter in the universe is in the plasma state; that is, in the form of an electrified gas with the atoms dissociated into positive ions and negative electrons [2]. This estimate may not be very accurate, but it is certainly a reasonable one in view of the fact that stellar interiors and atmospheres, gaseous nebulae, and much of the interstellar hydrogen are plasmas. In our own neighborhood, as soon as one leaves the earth's atmosphere, one encounters the plasma comprising the Van Allen radiation belts and the solar wind. On the other hand, in our everyday lives encounters with plasmas are limited to a few examples: the flash of a lightning bolt, the soft glow of the Aurora Borealis, the conducting gas inside a fluorescent tube or neon sign, and the slight amount of ionization in a rocket exhaust. It would seem that we live in the 1% of the universe in which plasmas do not occur naturally.

1.3 Debye Shielding:

A fundamental characteristic feature of a plasma is its ability to shield out electric potentials that are applied to it [3]. Suppose we tried to put an electric field inside a plasma by inserting two charged balls connected to a battery . The balls would attract particles of the opposite charge, and almost immediately a cloud of ions would surround the negative ball and a cloud of electrons would surround the positive ball. If the plasma were cold and there were no thermal motions, there would be just as many charges in the cloud as in the ball; the shielding would be perfect, and no electric field would be present in the body of the plasma outside of the clouds. On the other hand, if the temperature is finite, those particles that are at the edge of the cloud, where the electric field is weak, have enough thermal energy to escape from the electrostatic potential well. The "edge" of the cloud then occurs at the radius where the potential energy is approximately equal to the thermal energy KT of the particles, and the shielding is not complete. Potentials of the order of KT/e can leak into the plasma and cause finite electric fields to exist there.

2 Polarization of electromagnetic waves:

Polarization is a property applying to transverse waves that specifies the geometrical orientation of the oscillations. An electromagnetic wave such as light consists of a coupled oscillating electric field and magnetic field which are always perpendicular; by convention, the "polarization" of electromagnetic waves refers to the direction of the electric field. Polarization can be classified into three categories [4-7]:

- Linear Polarization - In electrodynamics, linear polarization or plane polarization of electromagnetic wave is a confinement of the electric field vector or magnetic field vector to a given plane along the direction of propagation. The orientation of a linearly polarized electromagnetic wave is defined by the direction of the electric field vector.For example, if the electric field vector is vertical (alternately up and down as the wave travels) the wave is said to be vertically polarized.
- Elliptical polarization - In electrodynamics, elliptical polarization is the polarization of electromagnetic wave such that the tip of the electric field vector describes an ellipse in any fixed plane intersecting, and normal to, the direction of propagation. An elliptically polarized wave may be resolved into two linearly polarized waves in phase quadrature, with their polarization planes at right angles to each other.
- Circular Polarization - In electrodynamics, circular polarization of an electromagnetic wave is a polarization state in which, at each point, the electric field of the wave has a constant magnitude but its direction rotates with time at a steady rate in a plane perpendicular to the direction of the wave. In electrodynamics the strength and direction of an electric field is defined by its electric field vector. In the case of a circularly polarized wave the tip of the electric field vector at a given point in space describes a circle as time progresses. At any instant of time, the electric field vector of the wave describes a helix along the direction of propagation. A circularly polarized wave can be in one of two possible states, right circular polarization in which the electric field vector rotates in a right-hand sense with respect to the direction of propagation, and left circular polarization in which the vector rotates in a left-hand sense.

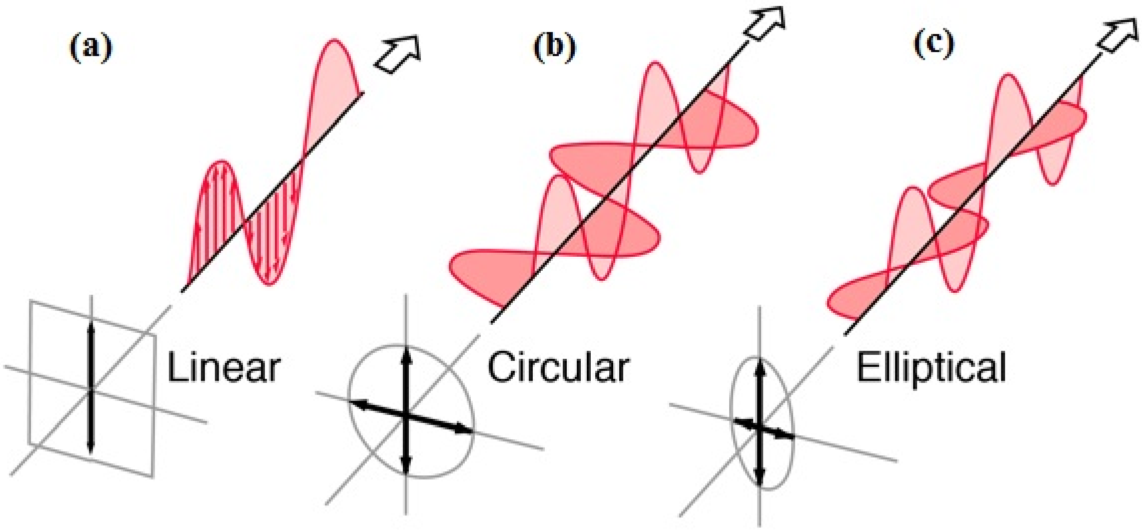

Figure 1: Polarization of electromagnetic waves is shown
(a) Linear Polarization
(b) Circular Polarization
(c) Elliptical Polarization

3 Basic Equations:

We consider the linear propagation of an electromagnetic wave in a collisional warm plasma which may be adequately described by the self-consistent solution of the following set of multi-fluid equations [1]:

The equation of motion for the species σ:

$$n_\sigma m_\sigma(\frac{\partial \mathbf{u}_\sigma}{\partial t} + \mathbf{u}_\sigma\cdot\nabla\mathbf{u}_\sigma) = n_\sigma q_\sigma(\mathbf{E}+ \mathbf{u}_\sigma\times\mathbf{B}) - \nabla p_\sigma - n_\sigma m_\sigma \nu_\sigma \mathbf{u}_\sigma \quad (1)$$

The equation of continuity:

$$\frac{\partial n_\sigma}{\partial t} + \nabla\cdot(n_\sigma\mathbf{u}_\sigma) = 0 \quad (2)$$

Maxwell's Equations:

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$

$$\nabla\times\mathbf{B} = \mu_0\sum n_\sigma q_\sigma \mathbf{u}_\sigma + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

The equation of state:

$$p_\sigma = n_\sigma T_\sigma \quad (5)$$

In Eqs.(1) in order to avoid unnecessary mathematical complications, we have taken a simple form for the collision term, namely - $n_\sigma m_\sigma \nu_\sigma \mathbf{u}_\sigma$. This collision term is appropriate, for example, to describe charged particle-neutral collisions in a weakly ionized plasma. We shall also assume that the

4

temperature T_σ of each species is constant and thus unaffected by the presence of the wave.

The equilibrium quantities , in the absence of the wave, are

$\mathbf{B}=B_0\hat{z}$, $n_\sigma=\overline{n_\sigma}$ (6)

where B_0 and $\overline{n_\sigma}$ are constants in space and time. The equilibrium values of $\mathbf{E}$ and $\mathbf{u}_\sigma$ are assumed to be zero.

We shall consider the propagation in the positive $\hat{z}$ direction (i.e. parallel to B_0) of a circularly polarized electromagnetic wave with an electric wave field of the form

$\mathbf{E}_\omega(z,t)= E_0(1,\pm i,0)\exp[iKz\text{-}i\omega t]$ (7)

where

$K = k(z) + i\gamma(z)$ $(k,\gamma$ real) (8)

and E_0 is the amplitude at z =0 which is constant in time. In Eqs.(7), the upper refers to right circularly polarized wave and lower sign refers to left circularly polarized wave.

From Eqs.(3) and Eqs.(7) we can obtain the magnetic field component $\mathbf{B}_\omega(\mathbf{z},\mathbf{t})$of the wave :

$$\nabla\times\mathbf{E}_\omega= -\frac{\partial\mathbf{B}_\omega}{\partial t}$$

$$\text{Now } \nabla\times\mathbf{E}_\omega=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_0exp(iKz-i\omega t) & \pm iE_0exp(iKz-i\omega t) & 0\end{vmatrix}$$

$$= \hat{x}\{0\pm i^2(K+z\tfrac{dK}{dz})E_0exp(iKz-i\omega t)\} - \hat{y}\{0-i(K+z\tfrac{dK}{dz})E_0exp(iKz-i\omega t)\} + \hat{z}\{0\text{-}0\}$$

$$=\{\pm(K+z\tfrac{dK}{dz})E_0exp(iKz-i\omega t)\}\hat{x}+\{i(K+z\tfrac{dK}{dz})E_0exp(iKz-i\omega t)\}\hat{y}+0\hat{z}$$

$$=(K+z\tfrac{dK}{dz})E_0(\pm1,i,0)exp(iKz-i\omega t)$$

Thus from Eqs.(3) we get

$$\frac{\partial\mathbf{B}_\omega}{\partial t} = -(K+z\tfrac{dK}{dz})E_0(\pm1,i,0)exp(iKz-i\omega t)$$

Now on integrating we get

$$\mathbf{B}_\omega(\mathbf{z},\mathbf{t}) =\frac{-(K+z\frac{dK}{dz})E_0(\pm1,i,0)exp(iKz-i\omega t)}{-i\omega} = \frac{-(K+z\frac{dK}{dz})E_0(\pm i,-1,0)exp(iKz-i\omega t)}{\omega}$$

Thus the magnetic field component is

$\mathbf{B}_\omega(\mathbf{z},\mathbf{t}) =\frac{1}{\omega}(K+z\frac{dK}{dz})E_0(\mp i,1,0)exp(iKz-i\omega t)$ (9)

We have $\mathbf{E}=\mathbf{E}_\omega+\mathbf{E_z}$ where $\mathbf{E}_\omega$is the perpendicular component and $\mathbf{E_z}$is the parallel component, and $\mathbf{B}$ $=B_0\hat{z}+\mathbf{B}_\omega$where $B_0\hat{z}$ is the parallel component and $\mathbf{B}_\omega$is the perpendicular component.

We can now split Eqs.(1) into its corresponding perpendicular and parallel components :

Perpendicular Component :

$$n_\sigma m_\sigma \frac{\partial \mathbf{u}_\sigma}{\partial t} = n_\sigma q_\sigma(\mathbf{E}_\omega + \mathbf{u}_\sigma \times \mathbf{B}_0) - n_\sigma m_\sigma \nu_\sigma \mathbf{u}_\sigma \quad (10)$$

Parallel Component :

$$n_\sigma q_\sigma(\mathbf{E_z} + \mathbf{u}_\sigma \times \mathbf{B}_\omega) - \frac{\partial p_\sigma}{\partial z} = 0 \quad (11)$$

Eqs.(10) is linear in wave quantities and thus we can obtain the fluid velocity for species σ.

We have from Eqs.(10)

$$n_\sigma m_\sigma \frac{\partial \mathbf{u}_\sigma}{\partial t} = n_\sigma q_\sigma(\mathbf{E}_\omega + \mathbf{u}_\sigma \times \mathbf{B}_0) - n_\sigma m_\sigma \nu_\sigma \mathbf{u}_\sigma$$

$$\Longrightarrow \frac{\partial \mathbf{u}_\sigma}{\partial t} = \frac{q_\sigma}{m_\sigma}E + \frac{\mathbf{q}_\sigma}{\mathbf{m}_\sigma}(\mathbf{u}_\sigma \times \mathbf{B}_0) - \nu_\sigma \mathbf{u}_\sigma$$

Denote $\frac{q_\sigma}{m_\sigma}B_0 = \Omega_\sigma$, which is the cyclotron frequency of the species σ.

Thus $$\frac{\partial \mathbf{u}_\sigma}{\partial t} = \frac{q_\sigma}{m_\sigma}E_0(1, \pm i, 0)exp[iKz - i\omega t] + \mathbf{u}_\sigma \times \boldsymbol{\Omega_\sigma} - \nu_\sigma \mathbf{u}_\sigma \quad (12)$$

We have $\mathbf{u}_\sigma = u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}}$ and $\boldsymbol{\Omega_\sigma} = \Omega_\sigma \hat{z}$

Now $$\mathbf{u}_\sigma \times \boldsymbol{\Omega_\sigma} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ u_x & u_y & 0 \\ 0 & 0 & \Omega_\sigma \end{vmatrix} = \hat{\mathbf{x}}(u_y\Omega_\sigma) - \hat{\mathbf{y}}(u_x\Omega_\sigma) + \hat{\mathbf{z}}.0$$

Now splitting Eqs.(12) in the direction of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ we obtain :

$$\frac{\partial u_x}{\partial t} = \frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] + u_y\Omega_\sigma - \nu_\sigma u_x \quad (13)$$

and

$$\frac{\partial u_y}{\partial t} = (\pm i)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - u_x\Omega_\sigma - \nu_\sigma u_y \quad (14)$$

Differentiating Eqs.(14) with respect to t we get

$$\frac{\partial^2 u_y}{\partial t^2} = (\pm i)(-i\omega)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - \Omega_\sigma\frac{\partial u_x}{\partial t} - \nu_\sigma\frac{\partial u_y}{\partial t}$$

$$\Longrightarrow \frac{\partial^2 u_y}{\partial t^2} = (\pm\omega)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - \Omega_\sigma\{\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] + u_y\Omega_\sigma - \nu_\sigma u_x\} - \nu_\sigma\frac{\partial u_y}{\partial t}$$

$$\Longrightarrow \frac{\partial^2 u_y}{\partial t^2} = (\pm\omega - \Omega_\sigma)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - u_y\Omega_\sigma^2 - \nu_\sigma u_x\Omega_\sigma - \nu_\sigma\frac{\partial u_y}{\partial t} \quad (15)$$

From Eqs.(14) we have :

$$\Omega_\sigma u_x = (\pm i)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - \nu_\sigma u_y - \frac{\partial u_y}{\partial t}$$

Substituting the value of $\Omega_\sigma u_x$ in Eqs.(15) we get

$$\frac{\partial^2 u_y}{\partial t^2} = (\pm\omega - \Omega_\sigma)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - u_y\Omega_\sigma^2 - \nu_\sigma\{(\pm i)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - \nu_\sigma u_y - \frac{\partial u_y}{\partial t}\} - \nu_\sigma\frac{\partial u_y}{\partial t}$$

$$\Longrightarrow \frac{\partial^2 u_y}{\partial t^2} = (\pm\omega - \Omega_\sigma \pm i\nu_\sigma)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t] - (\Omega_\sigma^2 + \nu_\sigma^2)u_y - 2\nu_\sigma\frac{\partial u_y}{\partial t}$$

$$\Longrightarrow (\frac{\partial^2}{\partial t^2} + \Omega_\sigma^2 + \nu_\sigma^2 + 2\nu_\sigma\frac{\partial}{\partial t})u_y = (\pm\omega - \Omega_\sigma \pm i\nu_\sigma)\frac{q_\sigma}{m_\sigma}E_0 exp[iKz - i\omega t]$$

Now for C.F. of u_y auxiliary equation is

$$m^2 + 2\nu_\sigma m + \nu_\sigma^2 + \Omega_\sigma^2 = 0$$

$$\Longrightarrow (m + \nu_\sigma)^2 = -\Omega_\sigma^2$$

$\Longrightarrow m + \nu_\sigma = \pm i\Omega_\sigma$

$\Longrightarrow m = -\nu_\sigma \pm i\Omega_\sigma$

$u_y = exp(-\nu_\sigma t)\{c_1 cos(\Omega_\sigma t) + c_2 sin(\Omega_\sigma t)\}$

Particular Integral :

$u_y = \frac{1}{\frac{\partial^2}{\partial t^2}+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma\frac{\partial}{\partial t}}(\pm\omega - \Omega_\sigma \pm i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$

$\Longrightarrow u_y = \frac{1}{i^2\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}(\pm\omega - \Omega_\sigma \pm i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$

$\Longrightarrow u_y = \frac{\mp 1}{-\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}(-\omega \pm \Omega_\sigma - i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$

Thus $u_y = exp(-\nu_\sigma t)\{c_1 cos(\Omega_\sigma t) + c_2 sin(\Omega_\sigma t)\} + \frac{\mp 1}{-\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}(-\omega \pm \Omega_\sigma - i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$ (16)

Differentiating Eqs.(13) w.r.t t we get

$\frac{\partial^2 u_x}{\partial t^2} = (-i\omega)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] + \Omega_\sigma\frac{\partial u_y}{\partial t} - \nu_\sigma\frac{\partial u_x}{\partial t}$

$\Longrightarrow \frac{\partial^2 u_x}{\partial t^2} = (-i\omega)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] + \Omega_\sigma\{(\pm i)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] - u_x\Omega_\sigma - \nu_\sigma u_y\} - \nu_\sigma\frac{\partial u_x}{\partial t}$

$\Longrightarrow \frac{\partial^2 u_x}{\partial t^2} = (-i\omega \pm i\Omega_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] - u_x\Omega_\sigma^2 - \nu_\sigma u_y\Omega_\sigma - \nu_\sigma\frac{\partial u_x}{\partial t}$ (17)

Now from Eqs.(13) we get

$u_y\Omega_\sigma = \frac{\partial u_x}{\partial t} - \frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] + \nu_\sigma u_x$

Substituting this value Eqs.(17) we get

$\frac{\partial^2 u_x}{\partial t^2} = (-i\omega \pm i\Omega_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] - u_x\Omega_\sigma^2 - \nu_\sigma\{\frac{\partial u_x}{\partial t} - \frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] + \nu_\sigma u_x\} - \nu_\sigma\frac{\partial u_x}{\partial t}$

$\Longrightarrow \frac{\partial^2 u_x}{\partial t^2} = (-i\omega \pm i\Omega_\sigma + i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t] - u_x\Omega_\sigma^2 - \nu_\sigma^2 u_x - 2\nu_\sigma\frac{\partial u_x}{\partial t}$

$\Longrightarrow (\frac{\partial^2}{\partial t^2} + \Omega_\sigma^2 + \nu_\sigma^2 + 2\nu_\sigma\frac{\partial}{\partial t})u_x = (-i\omega \pm i\Omega_\sigma + i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$

Now for C.F. of u_x auxiliary equation is

$m^2 + 2\nu_\sigma m + \nu_\sigma^2 + \Omega_\sigma^2 = 0$

$\Longrightarrow (m + \nu_\sigma)^2 = -\Omega_\sigma^2$

$\Longrightarrow m + \nu_\sigma = \pm i\Omega_\sigma$

$\Longrightarrow m = -\nu_\sigma \pm i\Omega_\sigma$

$u_x = exp(-\nu_\sigma t)\{c_3 cos(\Omega_\sigma t) + c_4 sin(\Omega_\sigma t)\}$

Particular Integral :

$u_x = \frac{1}{\frac{\partial^2}{\partial t^2}+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma\frac{\partial}{\partial t}}(-i\omega \pm i\Omega_\sigma + i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$

$\Longrightarrow u_x = \frac{1}{i^2\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}(-i\omega \pm i\Omega_\sigma + i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$

$\Longrightarrow u_x = \frac{i}{-\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}(-\omega \pm \Omega_\sigma - i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz - i\omega t]$

Thus $u_x = exp(-\nu_\sigma t)\{c_3 cos(\Omega_\sigma t) + c_4 sin(\Omega_\sigma t)\} + \frac{i}{-\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}(-\omega \pm \Omega_\sigma - i\nu_\sigma)\frac{q_\sigma}{m_\sigma}\mathrm{E}_0 exp[iKz -$

$i\omega t]$ (18)

Thus from Eqs.(16) and (18) we obtain the fluid velocity for species σ :

$u_\sigma(z,t) = \frac{(-\omega\pm\Omega_\sigma-i\nu_\sigma)}{-\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}\frac{q_\sigma}{m_\sigma}\mathrm{E}_0(i,\mp1,0)exp[iKz-i\omega t]$

Now $\frac{(-\omega\pm\Omega_\sigma-i\nu_\sigma)}{-\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)}$

$$= \frac{(-\omega\pm\Omega_\sigma-i\nu_\sigma)}{\Omega_\sigma^2-(\omega+i\nu_\sigma)^2} = \begin{cases} \frac{(-\omega+\Omega_\sigma-i\nu_\sigma)}{(\Omega_\sigma-\omega-i\nu_\sigma)(\Omega_\sigma+\omega+i\nu_\sigma)} = \frac{1}{\Omega_\sigma+\omega+i\nu_\sigma} \\ \frac{(-\omega-\Omega_\sigma-i\nu_\sigma)}{(\Omega_\sigma-\omega-i\nu_\sigma)(\Omega_\sigma+\omega+i\nu_\sigma)} = \frac{1}{-\Omega_\sigma+\omega+i\nu_\sigma} \end{cases}$$

Thus $\frac{(-\omega\pm\Omega_\sigma-i\nu_\sigma)}{-\omega^2+\Omega_\sigma^2+\nu_\sigma^2+2\nu_\sigma(-i\omega)} = \frac{1}{\omega\pm\Omega_\sigma+i\nu_\sigma}$

Thus $u_\sigma(z,t) = exp(-\nu_\sigma t)\{(c_1,c_3,0)cos(\Omega_\sigma t)+(c_2,c_4,0)sin(\Omega_\sigma t)\}+\frac{1}{\omega\pm\Omega_\sigma+i\nu_\sigma}\frac{q_\sigma}{m_\sigma}\mathrm{E}_0(i,\mp1,0)exp[iKz-$
$i\omega t]$ (19)

Discarding the exponential decay part of the velocity of Eqs.(19) we obtain

$u_\sigma(z,t) = \frac{1}{\omega\pm\Omega_\sigma+i\nu_\sigma}\frac{q_\sigma}{m_\sigma}\mathrm{E}_0(i,\mp1,0)exp[iKz-i\omega t]$ (20)

The force balance along the direction of the magnetic field $\mathbf{B_0}$is given by Eqs.(11). Now taking the real part of Eqs.(9) and Eqs.(12) , it may be shown that for a circularly polarized wave propagating in a collisional plasma, the nonlinear(ponderomotive) force exerted by the wave on each species is independent of time.

From Eqs.(8) and (20) we have

$u_\sigma(z,t) = \frac{1}{\omega\pm\Omega_\sigma+i\nu_\sigma}\frac{q_\sigma}{m_\sigma}\mathrm{E}_0(i,\mp1,0)exp[-z\gamma+i(zk-\omega t)]$

$\Longrightarrow u_\sigma(z,t) = \frac{\omega\pm\Omega_\sigma-i\nu_\sigma}{(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2}\frac{q_\sigma}{m_\sigma}exp(-z\gamma)\mathrm{E}_0(i,\mp1,0)\{cos(zk-\omega t)+isin(zk-\omega t)\}$

$\Longrightarrow u_\sigma(z,t) = \frac{q_\sigma exp(-z\gamma)}{m_\sigma\{(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2\}}\mathrm{E}_0(i(\omega\pm\Omega_\sigma)+\nu_\sigma,\mp(\omega\pm\Omega_\sigma)\pm i\nu_\sigma,0)\{cos(zk-\omega t)+isin(zk-\omega t)\}$

Thus real part of

$u_\sigma(z,t)= \frac{q_\sigma exp(-z\gamma)}{m_\sigma\{(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2\}}\mathrm{E}_0[-(\omega\pm\Omega_\sigma)sin(zk-\omega t)+\nu_\sigma cos(zk-\omega t),\mp(\omega\pm\Omega_\sigma)cos(zk-\omega t)\mp\nu_\sigma sin(zk-$
$\omega t),0]$ (21)

From Eqs.(8) and Eqs.(9) we have

$\mathbf{B}_\omega(\mathbf{z},\mathbf{t}) =\frac{1}{\omega}(k+i\gamma+z\frac{dk}{dz}+iz\frac{d\gamma}{dz})E_0(\mp i,1,0)exp(-z\gamma+i(zk-\omega t))$

$\Longrightarrow\mathbf{B}_\omega(\mathbf{z},\mathbf{t}) =\frac{1}{\omega}\{(k+z\frac{dk}{dz})+i(\gamma+z\frac{d\gamma}{dz})\}E_0(\mp i,1,0)exp(-z\gamma)[cos(zk-\omega t)+isin(zk-\omega t)]$

$\Longrightarrow\mathbf{B}_\omega(\mathbf{z},\mathbf{t}) =\frac{1}{\omega}\{(k+z\frac{dk}{dz})+i(\gamma+z\frac{d\gamma}{dz})\}E_0exp(-z\gamma)[\mp icos(zk-\omega t)\pm sin(zk-\omega t),cos(zk-\omega t)+$
$isin(zk-\omega t),0]$

Thus real part of

$\mathbf{B}_\omega(\mathbf{z},\mathbf{t}) =\frac{exp(-z\gamma)}{\omega}E_0[\pm(k+z\frac{dk}{dz})sin(zk-\omega t)\pm(\gamma+z\frac{d\gamma}{dz})cos(zk-\omega t),\ (k+z\frac{dk}{dz})cos(zk-\omega t)-(\gamma+$

8

$z\frac{d\gamma}{dz})sin(zk-\omega t),\,0]$ (22)

Using Eqs.(21) and Eqs.(22) we have

$$\mathbf{u}_\sigma \times \mathbf{B}_\omega = \frac{q_\sigma exp(-2z\gamma)E_0^2}{m_\sigma \omega\{(\omega \pm \Omega_\sigma)^2+\nu_\sigma^2\}} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -(\omega \pm \Omega_\sigma)sin(zk-\omega t)+\nu_\sigma cos(zk-\omega t) & \mp(\omega \pm \Omega_\sigma)cos(zk-\omega t) \mp \nu_\sigma sin(zk-\omega t) & 0 \\ \pm(k+z\frac{dk}{dz})sin(zk-\omega t) \pm (\gamma + z\frac{d\gamma}{dz})cos(zk-\omega t) & (k+z\frac{dk}{dz})cos(zk-\omega t)-(\gamma+z\frac{d\gamma}{dz})sin(zk-\omega t) & 0 \end{vmatrix}$$

$\implies \mathbf{u}_\sigma \times \mathbf{B}_\omega = \hat{\mathbf{z}}\frac{q_\sigma exp(-2z\gamma)E_0^2}{m_\sigma \omega\{(\omega \pm \Omega_\sigma)^2+\nu_\sigma^2\}}[-(\omega \pm \Omega_\sigma)(k+z\frac{dk}{dz})sin(zk-\omega t)cos(zk-\omega t)+(\omega \pm \Omega_\sigma)(\gamma + z\frac{d\gamma}{dz})sin^2(zk-\omega t)+\nu_\sigma(k+z\frac{dk}{dz})cos^2(zk-\omega t)-\nu_\sigma(\gamma+z\frac{d\gamma}{dz})sin(zk-\omega t)cos(zk-\omega t)+(\omega \pm \Omega_\sigma)(k+z\frac{dk}{dz})sin(zk-\omega t)cos(zk-\omega t)+\nu_\sigma(k+z\frac{dk}{dz})sin^2(zk-\omega t)+\nu_\sigma(\gamma+z\frac{d\gamma}{dz})sin(zk-\omega t)cos(zk-\omega t)+(\omega \pm \Omega_\sigma)(\gamma+z\frac{d\gamma}{dz})cos^2(zk-\omega t)]$

$\implies \mathbf{u}_\sigma \times \mathbf{B}_\omega = \hat{\mathbf{z}}\frac{q_\sigma exp(-2z\gamma)E_0^2}{m_\sigma \omega\{(\omega \pm \Omega_\sigma)^2+\nu_\sigma^2\}}[(\omega \pm \Omega_\sigma)(\gamma+z\frac{d\gamma}{dz})+\nu_\sigma(k+z\frac{dk}{dz})]$

$\implies \mathbf{u}_\sigma \times \mathbf{B}_\omega = \hat{\mathbf{z}}\frac{q_\sigma exp(-2z\gamma)E_0^2}{m_\sigma \omega\{(\omega \pm \Omega_\sigma)^2+\nu_\sigma^2\}}\frac{d}{dz}\{\nu_\sigma kz + (\omega \pm \Omega_\sigma)\gamma z\}$

Thus the nonlinear (ponderomotive) force exerted by the wave on each species is independent of time and given by

$$q_\sigma(\mathbf{u}_\sigma \times \mathbf{B}_\omega) = \hat{\mathbf{z}}\frac{q_\sigma^2 exp(-2z\gamma)E_0^2}{m_\sigma \omega\{(\omega \pm \Omega_\sigma)^2+\nu_\sigma^2\}}\frac{d}{dz}\{\nu_\sigma kz + (\omega \pm \Omega_\sigma)\gamma z\} \quad (23)$$

In the steady state, the ponderomotive force is balanced by the axial pressure gradients and the time-independent electrostatic field E_z, which arise from the spatial separation of the different species.

The electric wave field given by Eqs.(7) must satisfy the wave equation obtained from Eqs.(3) and Eqs.(4) :

Differentiating Eqs.(4) partially w.r.t t we have

$$\nabla \times \frac{\partial \mathbf{B}_\omega}{\partial t} = \mu_0 \sum n_\sigma q_\sigma \frac{\partial u_\sigma}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_\omega}{\partial t^2} \quad (24)$$

Now from Eqs.(3) and Eqs.(24) we have

$$-\nabla \times \nabla \times \mathbf{E}_\omega = \mu_0 \sum n_\sigma q_\sigma \frac{\partial u_\sigma}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_\omega}{\partial t^2}$$

$$\implies \nabla^2 \mathbf{E}_\omega - \nabla(\nabla.\mathbf{E}_\omega) = \mu_0 \sum n_\sigma q_\sigma \frac{\partial u_\sigma}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_\omega}{\partial t^2} \quad (25)$$

Now $\nabla.\mathbf{E}_\omega = \mathbf{0}$and $\nabla^2 \mathbf{E}_\omega = \frac{\partial^2 \mathbf{E}_\omega}{\partial z^2}$. So from Eqs.(25) we have

$$\frac{\partial^2 \mathbf{E}_\omega}{\partial z^2} = \mu_0 \sum n_\sigma q_\sigma \frac{\partial u_\sigma}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_\omega}{\partial t^2} \quad (26)$$

Differentiating Eqs.(7) partially w.r.t z we have :

$$\frac{\partial \mathbf{E}_\omega}{\partial z} = E_0(1, \pm i, 0)exp(iKz - i\omega t)[i(K + z\frac{dK}{dz})]$$

$$\frac{\partial^2 \mathbf{E}_\omega}{\partial z^2} = E_0(1, \pm i, 0)exp(iKz - i\omega t)[-(K + z\frac{dK}{dz})^2 + i(2\frac{dK}{dz} + z\frac{d^2K}{dz^2})]$$

9

$\Longrightarrow \frac{\partial^2 \mathbf{E}_\omega}{\partial z^2} = \mathbf{E}_\omega[-(K + z\frac{dK}{dz})^2 + i(2\frac{dK}{dz} + z\frac{d^2K}{dz^2})]$ (27)

Differentiating Eqs.(7) partially w.r.t t we have :

$\frac{\partial \mathbf{E}_\omega}{\partial t} = E_0(1, \pm i, 0)exp(iKz - i\omega t)[-i\omega]$

$\frac{\partial^2 \mathbf{E}_\omega}{\partial t^2} = E_0(1, \pm i, 0)exp(iKz - i\omega t)[-i\omega]^2$

$\Longrightarrow \frac{\partial^2 \mathbf{E}_\omega}{\partial t^2} = -\mathbf{E}_\omega \omega^2$ (28)

Differentiating Eqs.(20) partially w.r.t t we have :

$\frac{\partial u_\sigma}{\partial t} = \frac{1}{\omega \pm \Omega_\sigma + i\nu_\sigma}\frac{q_\sigma}{m_\sigma}\mathrm{E}_0(i, \mp 1, 0)exp[iKz - i\omega t][-i\omega]$

$\Longrightarrow \frac{\partial u_\sigma}{\partial t} = \frac{1}{\omega \pm \Omega_\sigma + i\nu_\sigma}\frac{q_\sigma}{m_\sigma}\omega \mathrm{E}_0(1, \pm i, 0)exp[iKz - i\omega t]$

$\Longrightarrow \frac{\partial u_\sigma}{\partial t} = \frac{1}{\omega \pm \Omega_\sigma + i\nu_\sigma}\frac{q_\sigma}{m_\sigma}\omega \mathbf{E}_\omega$ (29)

Thus from Eqs.(26),(27),(28) and (29) we have

$\mathbf{E}_\omega[-(K + z\frac{dK}{dz})^2 + i(2\frac{dK}{dz} + z\frac{d^2K}{dz^2})] = \mu_0 \sum \frac{1}{\omega \pm \Omega_\sigma + i\nu_\sigma}\frac{n_\sigma q_\sigma^2}{m_\sigma}\omega \mathbf{E}_\omega - \frac{\omega^2}{c^2}\mathbf{E}_\omega$

$\Longrightarrow -(K + z\frac{dK}{dz})^2 + i(2\frac{dK}{dz} + z\frac{d^2K}{dz^2}) = \mu_0 \omega^2 \sum \frac{1}{\omega(\omega \pm \Omega_\sigma + i\nu_\sigma)}\frac{n_\sigma q_\sigma^2}{m_\sigma} - \frac{\omega^2}{c^2}$

$\Longrightarrow (K + z\frac{dK}{dz})^2 - i(2\frac{dK}{dz} + z\frac{d^2K}{dz^2}) - \frac{\omega^2}{c^2}\{1 - c^2\mu_0 \sum \frac{1}{\omega(\omega \pm \Omega_\sigma + i\nu_\sigma)}\frac{n_\sigma q_\sigma^2}{m_\sigma}\} = 0$

$\Longrightarrow K^2 + (z\frac{dK}{dz})^2 + 2Kz\frac{dK}{dz} - 2i\frac{dK}{dz} - iz\frac{d^2K}{dz^2} - \frac{\omega^2}{c^2}\{1 - \mu_0 c^2 \sum \frac{1}{\omega(\omega \pm \Omega_\sigma + i\nu_\sigma)}\frac{n_\sigma q_\sigma^2}{m_\sigma}\} = 0$

$\Longrightarrow K^2 + 2Kz\frac{dK}{dz} - 2i\frac{dK}{dz} + (z\frac{dK}{dz})^2 - iz\frac{d^2K}{dz^2} - \frac{\omega^2}{c^2}\{1 - \sum \frac{1}{\omega(\omega \pm \Omega_\sigma + i\nu_\sigma)}\frac{n_\sigma q_\sigma^2}{m_\sigma \epsilon_0}\} = 0$ (since $\mu_0 \epsilon_0 = \frac{1}{c^2}$)

$\Longrightarrow K^2 + 2Kz\frac{dK}{dz} - 2i\frac{dK}{dz} + (z\frac{dK}{dz})^2 - iz\frac{d^2K}{dz^2} - \frac{\omega^2}{c^2}\{1 - \sum \frac{\omega_{p\sigma}^2}{\omega(\omega \pm \Omega_\sigma + i\nu_\sigma)}\} = 0$ (30)

Equation (30) is the nonlinear dispersion relation and $\omega_{p\sigma} = \sqrt{\frac{n_\sigma q_\sigma^2}{m_\sigma \epsilon_0}}$ is the plasma frequency for the species σ.

4 The linear wavenumber :

If we assume that the wave amplitude is sufficiently small to neglect the nonlinear interaction, the wavenumber (denoted as $K_0 = k_0 + i\gamma_0$ for the linear solution) is independent of z and given , using Eqs.(30), by

$K_0^2 - \frac{\omega^2}{c^2}\{1 - \sum \frac{\omega_{p\sigma}^2}{\omega(\omega \pm \Omega_\sigma + i\nu_\sigma)}\} = 0$ (31)

This is the usual form of the linear dispersion relation for an electromagnetic wave propagating in a collisional plasma(Ginzburg,1961)

Define

$A = 1 - \sum \frac{\omega_{p\sigma}^2(\omega \pm \Omega_\sigma)}{\omega[(\omega \pm \Omega_\sigma)^2 + \nu_\sigma^2]}$ (32)

and

10

$B=\sum\frac{\omega_{p\sigma}^2\nu_\sigma}{\omega[(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2]}$ (33)

From (8) we have $K_0^2=k_0^2-\gamma_0^2+2ik_0\gamma_0$

Now $1-\sum\frac{\omega_{p\sigma}^2}{\omega(\omega\pm\Omega_\sigma+i\nu_\sigma)}=1-\sum\frac{\omega_{p\sigma}^2(\omega\pm\Omega_\sigma-i\nu_\sigma)}{\omega[(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2]}=1-\sum\frac{\omega_{p\sigma}^2(\omega\pm\Omega_\sigma)}{\omega[(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2]}+i\sum\frac{\omega_{p\sigma}^2\nu_\sigma}{\omega[(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2]}=A+iB$

Thus $K_0^2=\frac{\omega^2}{c^2}[A+iB]$

$\implies k_0^2-\gamma_0^2+2ik_0\gamma_0=\frac{\omega^2}{c^2}[A+iB]$

Equating real and imaginary parts we have :

Real Part :

$k_0^2-\gamma_0^2=\frac{\omega^2}{c^2}A$

$\implies k_0^2=\gamma_0^2+\frac{\omega^2}{c^2}A$ (34)

Imaginary part:

$2k_0\gamma_0=\frac{\omega^2}{c^2}B$

$\implies 4k_0^2\gamma_0^2=\frac{\omega^4}{c^4}B^2$ (35)

Now substituting the value of $k_0^2=\gamma_0^2+\frac{\omega^2}{c^2}A$ in Eqs.(35) we have

$\implies 4[\gamma_0^2+\frac{\omega^2}{c^2}A]\gamma_0^2=\frac{\omega^4}{c^4}B^2$

$\implies 4\gamma_0^4+\frac{4\omega^2}{c^2}A\gamma_0^2-\frac{\omega^4}{c^4}B^2=0$

Solving for γ_0^2we have

$\gamma_0^2=\left[-\frac{4\omega^2}{c^2}A\pm\sqrt{\frac{16\omega^4}{c^4}A^2+16\frac{\omega^4}{c^4}B^2}\right]/8$

$\implies \gamma_0^2=\frac{1}{8}\left[-\frac{4\omega^2}{c^2}A\pm\frac{4\omega^2}{c^2}\sqrt{A^2+B^2}\right]$

$\implies \gamma_0^2=\frac{\omega^2}{2c^2}\left[-A\pm\sqrt{A^2+B^2}\right]$

Thus

$\gamma_0^2=\frac{\omega^2}{2c^2}\left[-A+\sqrt{A^2+B^2}\right]$ (36)

Now

$k_0^2=\gamma_0^2+\frac{\omega^2}{c^2}A$

$\implies k_0^2=\frac{\omega^2}{2c^2}\left[-A+\sqrt{A^2+B^2}\right]+\frac{\omega^2}{c^2}A$

$\implies k_0^2=\frac{\omega^2}{2c^2}\left[A+\sqrt{A^2+B^2}\right]$ (37)

In the presence of collisions, the wave does not suffer from the effects of resonance and cut-off; for all frequencies the wave possesses a non-zero,finite value of k_0. However, in the frequency ranges of cut-off for the undamped wave, we find $\gamma_0\gg k_0$,that is, the wave is heavily damped.

We note that the ratio of real and imaginary parts of the complex wavenumber may be written as

$$\frac{k_0}{\gamma_0} = \frac{\sqrt{A+\sqrt{A^2+B^2}}}{\sqrt{-A+\sqrt{A^2+B^2}}} = \frac{A+\sqrt{A^2+B^2}}{\sqrt{-A^2+A^2+B^2}} = \frac{A}{B} + \sqrt{1+\frac{A^2}{B^2}} \quad (38)$$

If the collision frequency ν_σ is sufficiently small,then for $\omega \backsimeq \pm\Omega_\sigma$we may approximate A and B by the contribution due to the species σ alone. Thus we have

$A \backsimeq -\frac{\omega_{p\sigma}^2(\omega\pm\Omega_\sigma)}{\omega[(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2]}$ and $B \backsimeq \frac{\omega_{p\sigma}^2\nu_\sigma}{\omega[(\omega\pm\Omega_\sigma)^2+\nu_\sigma^2]}$

and substituting them into Eqs.(38) we have

$$\frac{k_0}{\gamma_0} \backsimeq \frac{-(\omega\pm\Omega_\sigma)}{\nu_\sigma} + \sqrt{1+\frac{(\omega\pm\Omega_\sigma)^2}{\nu_\sigma^2}}$$

$$\Longrightarrow \frac{\nu_\sigma k_0}{\gamma_0} \backsimeq -(\omega\pm\Omega_\sigma) + \sqrt{\nu_\sigma^2+(\omega\pm\Omega_\sigma)^2} \quad (39)$$

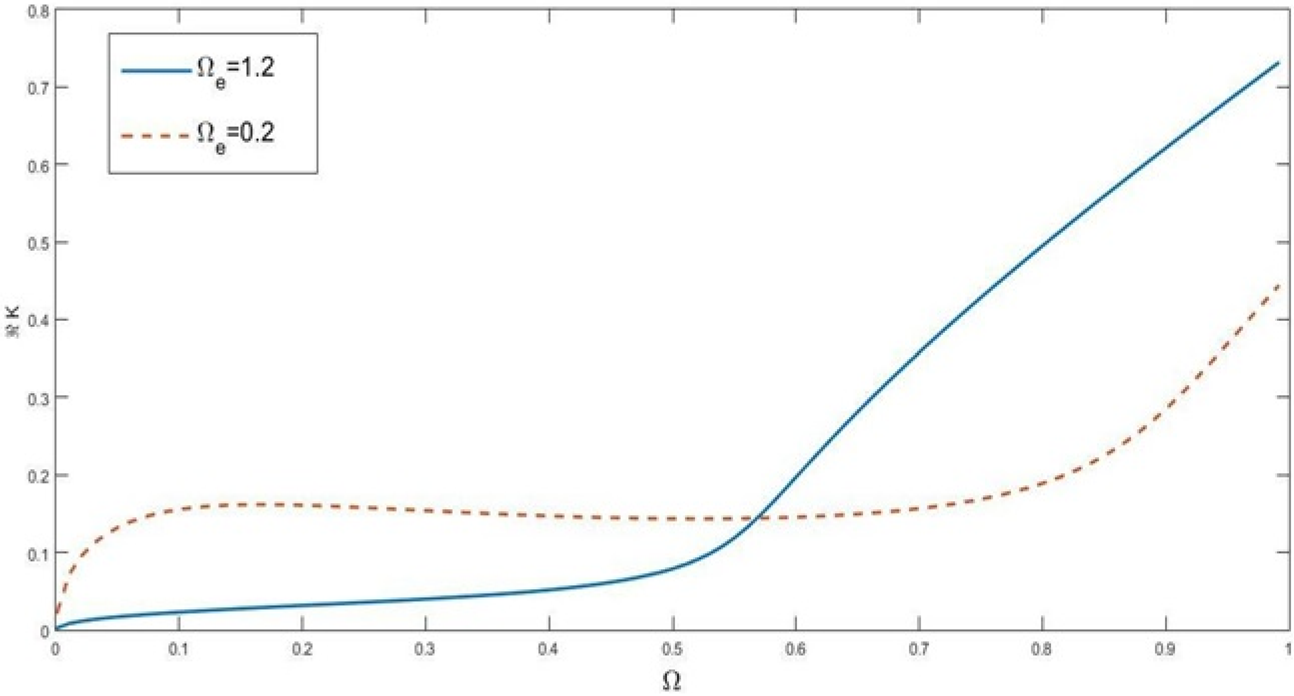

Figure 2: The real part of K is plotted against the wave frequency for different values of ′Ωe =1.2(solid line), 0.2(dotted line) for RCP wave.

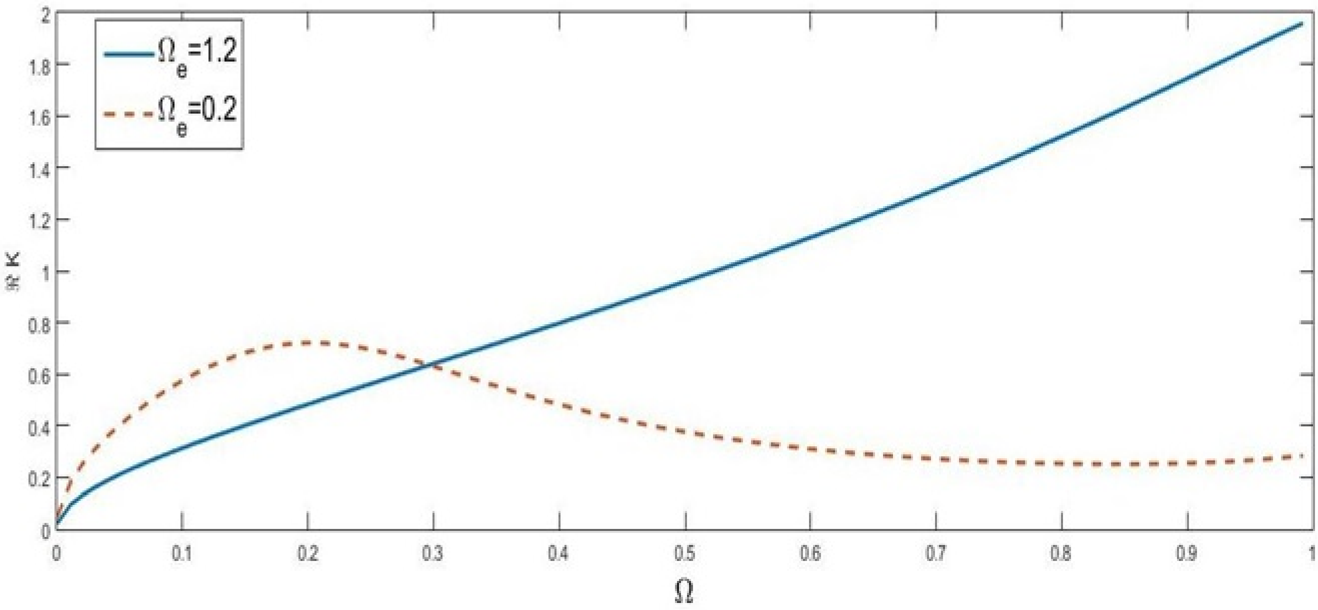

Figure 3: The real part of K is plotted against the wave frequency for different values of ΄Ωe =1.2(solid line), 0.2(dotted line) for LCP wave.

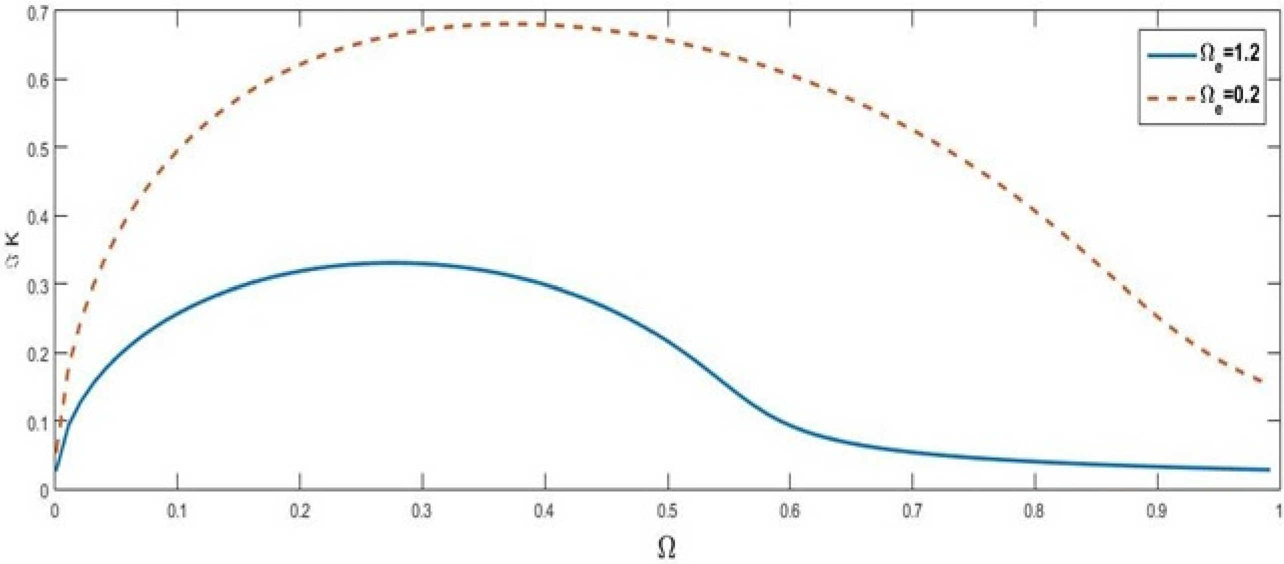

Figure 4: The imaginary part of K is plotted against the wave frequency for different values of ΄Ωe =1.2(solid line), 0.2(dotted line) for RCP wave.

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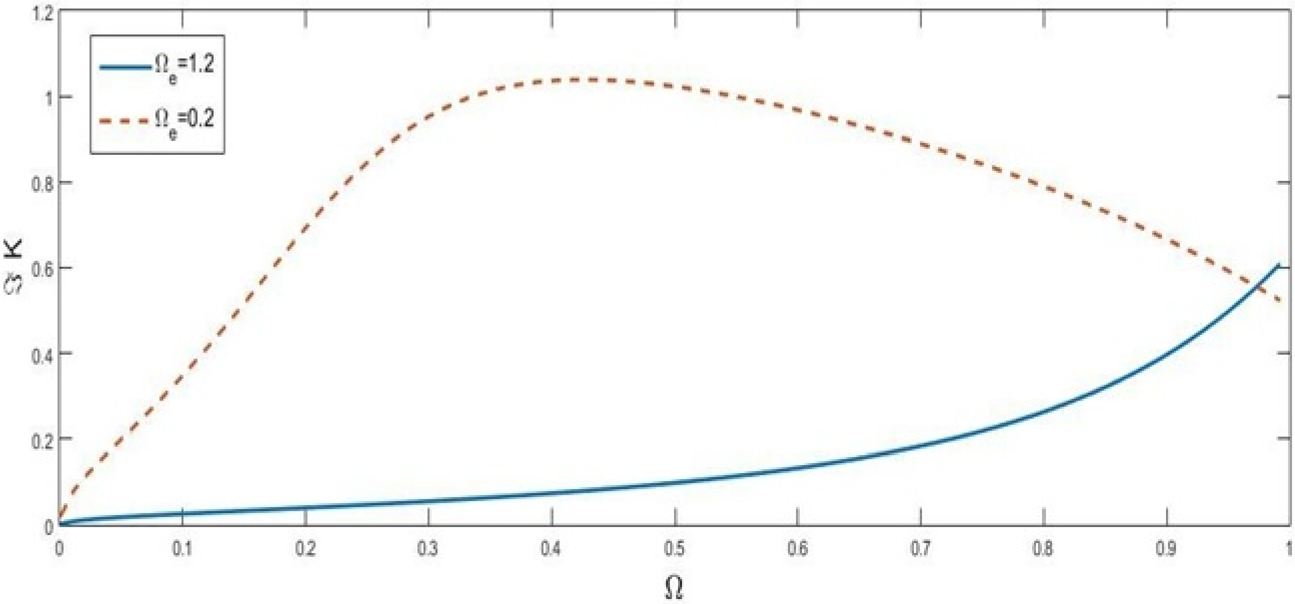

Figure 5: The imaginary part of K is plotted against the wave frequency for different values of Ωe =1.2(solid line), 0.2(dotted line) for LCP wave

5 Conclusion:

We have studied the propagation characteristics of an intense circularly polarized electromagnetic wave in a collisional warm plasma. Starting from a set of hydrodynamic equations coupled to the Maxwell equations we have obtained the dispersion relation for the electromagnetic waves. The properties of the modes together with the damping rate are analysed analytically and numerically with different plasma parameters.

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